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**HAL Id: halshs-00119033**

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Submitted on 7 Dec 2006

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**2006.66**



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<http://mse.univ-paris1.fr/Publicat.htm>

ISSN : 1624-0340

# Some fixed point theorems for discontinuous mappings

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**Abstract** This paper provides a fixed point theorem à la Schauder, where the mappings considered are possibly discontinuous. Our main result generalizes and unifies several well-known results.

**Résumé:** Nous prouvons dans cet article un théorème de point fixe du type Schauder (dont un corollaire en dimension finie est le théorème de Brouwer) pour des applications qui peuvent présenter des discontinuités. Notre résultat principal généralise et unifie de nombreux résultats connus.

*AMS classification:* 47H10

*Key words:* Schauder fixed point theorem, Brouwer fixed point theorem, discontinuity.

## 1. INTRODUCTION

From one version of Schauder fixed point theorem, every continuous mapping  $f : C \rightarrow C$ , where  $C$  is a compact and convex subset of a Banach space  $E$ , has a fixed point. A corollary of particular importance, which allows to derive many existence results in nonlinear analysis, is Brouwer fixed point theorem, which corresponds to the case where  $C$  is a compact and convex subset of  $\mathbb{R}^n$ .

Yet, in many applications, it would be interesting to relax the continuity assumption. Several papers have already extended Schauder fixed point theorem, or Brouwer fixed point theorem, to noncontinuous mappings: see, for example, [3], [5], [6] or [8].<sup>2</sup>

Roughly, several of these papers have proved that approximate continuous mappings admit approximate fixed points. The main techniques used to prove such theorems are the approximation of the discontinuous function by a continuous one (see, for example, [8]) or the extension of the discontinuous mapping to a multivalued mapping which admits fixed points (see, for example, [5]).

The aim of this paper is to provide a new type of discontinuity (which we call half-continuity), and to prove that Schauder theorem can be extended to the case of half-continuous mappings. Our approach has several advantages:

- (1) Contrary to approximate continuity, half-continuity allows large discontinuities.

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<sup>1</sup>I wish to thank Moncef Meddeb and Julie Caporiccio for valuable remarks. Errors are mine.

<sup>2</sup>Another class of fixed point theorems allows noncontinuous mappings, the class of order theoretic fixed point results. But it requires monotonicity properties, which we do not require in this paper.

- (2) The class of fixed point theorems obtained contains the standard fixed point theorems as Brouwer's one or Schauder's one.
- (3) Our approach easily implies, and sometimes improves, some standard approximate fixed point theorems (as [8] and [3]).
- (4) Our main fixed point result admits an immediate generalization to multivalued mappings, which generalizes Kakutani's theorem, and under standard assumptions on the multivalued mappings, any selection without fixed points is half-continuous.
- (5) A consequence of our main theorem is an answer to an open question raised by [10].

The paper is organized as follows. In Section 2, one gives the definition of half-continuity, and relate it to other weak continuity notions. In Section 3, one states and proves the main discontinuous fixed point theorem, with some immediate applications. In Section 4, one shows that our main result generalizes some approximated fixed point theorems as Bula theorem (see [3]) or Klee Theorem (see [8]).

## 2. HALF-CONTINUOUS MAPPINGS

In this section, we define the main property of this paper, which we call half-continuity. Then we present some examples of half-continuous mappings which are not continuous.

First of all, let us precise our notations: in the following, for every Banach space  $(E, \|\cdot\|)$ ,  $E^*$  denotes the topological dual of  $E$ , endowed with the strong norm. For every  $p \in E^*$  and  $x \in E$ , one denotes  $\langle p, x \rangle$  the value of  $p$  at  $x$ , i.e.  $p(x)$ . Besides, for every  $x \in E$  and  $r \geq 0$ ,  $B(x, r) = \{y \in E, \|y - x\| \leq r\}$  is the closed ball of radius  $r$  centered at  $x$ . For every closed subset  $K$  of a Banach space  $E$  and every  $x \in E$ ,  $d_K(x)$  denotes the standard distance from  $x$  to  $K$ , defined by  $d_K(x) = \inf_{y \in K} \|y - x\|$ . For every subsets  $C$  of a set  $E$ , for every mapping  $f : C \rightarrow E$  (respectively for every multivalued mapping  $F$  from  $C$  to  $E$ ) one defines  $Fix(f) = \{x \in C, f(x) = x\}$  (respectively  $Fix(F) = \{x \in C, x \in F(x)\}$ ) the set of fixed points of  $f$  (respectively of  $F$ ). Lastly, all the sets in this paper are supposed to be nonempty

**Definition 2.1.** Let  $E$  be a Banach space, and let  $C \subset E$ . A mapping  $f : C \rightarrow E$  is said to be half-continuous if:

$$\forall x \in C, x \neq f(x) \Rightarrow \exists p \in E^*, \exists \epsilon > 0 \text{ such that:}$$

$$\forall x' \in B(x, \epsilon) \cap C, x' \neq f(x') \Rightarrow \langle p, f(x') - x' \rangle > 0.$$

*Remark 2.2.* Roughly,  $f$  is a half-continuous mapping if every  $x$  which is not a fixed point of  $f$  can be separated from its image  $f(x)$ , the separation being robust to a small perturbation of  $x$ .

*Remark 2.3.* In the case where  $E = \mathbb{R}$ , half-continuity simply means that  $f(x) - x$  has constant sign locally around every  $x \in E$  satisfying  $f(x) - x \neq 0$ .

The following proposition makes a link between half-continuity and lower (respectively upper) semi-continuity. In this proposition, for every  $p \in E^*$ ,  $\phi_p : C \rightarrow \mathbb{R}$  denotes the mapping defined by  $\phi_p(x) = \langle p, f(x) \rangle$  for every  $x \in C$ .

**Proposition 2.4.** Let  $E$  be a Banach space and  $C \subset E$ . Consider a mapping  $f : C \rightarrow E$  such that:

$\forall x \in C, x \neq f(x) \Rightarrow \exists p \in E^*$  such that (i) or (ii) is true:

(i)  $\langle p, f(x) - x \rangle > 0$  and  $\phi_p$  is lower semi-continuous at  $x$ .

(ii)  $\langle p, f(x) - x \rangle < 0$  and  $\phi_p$  is upper semi-continuous at  $x$ .

Then  $f$  is half-continuous.

*Proof.* Let  $f : C \rightarrow E$  which satisfies the assumptions of the proposition above. Let  $x \in C$  with  $f(x) \neq x$ , and suppose, for example, that  $\langle p, f(x) - x \rangle > 0$  and that  $\phi_p$  is lower semi-continuous at  $x$  for some  $p \in E^*$ . Let  $\epsilon > 0$  such that

$$(2.1) \quad \eta = \langle p, f(x) - x \rangle + \min_{x' \in B(x, \epsilon)} \langle p, x - x' \rangle > 0.$$

Since  $\phi_p$  is lower semi-continuous at  $x$ , there exists  $\epsilon' > 0$  such that for every  $x' \in B(x, \epsilon')$ , one has

$$(2.2) \quad \langle p, f(x') \rangle > \langle p, f(x) \rangle - \eta.$$

Now, for every  $x' \in B(x, \min\{\epsilon, \epsilon'\})$ , one has

$$(2.3) \quad \langle p, f(x') - x' \rangle > \langle p, f(x) - x \rangle - \eta + \langle p, x - x' \rangle > 0$$

the first inequality being a consequence of Inequality 2.2, and the second inequality a consequence of Equation 2.1, which defines  $\eta$ . This ends the proof of the proposition.  $\square$

*Remark 2.5.* In Proposition 2.4, the converse implication is not true. For example, let define the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = 3$  if  $x \in [0, 1[$  and  $f(x) = 2$  elsewhere. It is clearly half-continuous, but at  $x = 0$ , it cannot satisfy (i) or (ii) of Proposition 2.4.

If  $E$  is a Hilbert space, then the previous proposition implies:

**Proposition 2.6.** Let  $H$  be a Hilbert space, and let  $C \subset H$ . Consider a mapping  $f : C \rightarrow H$  such that:

$\forall x \in C, x \neq f(x) \Rightarrow \phi_p(\cdot)$  is lower semi-continuous at  $x$ , where  $p = f(x) - x$ .

Then  $f$  is half-continuous.

We now provide another examples of half-continuous mappings which are not continuous.

**Example 2.7.** Let  $E$  denotes a Banach space, endowed with the strong norm  $\|\cdot\|$ , and let  $E_w$  be the space  $E$  endowed with the weak topology. Let  $f : E \rightarrow E_w$  be a continuous mapping, which means that for every  $p \in E^*$ , the mapping  $\phi_p : E \rightarrow \mathbb{R}$  defined by  $\forall x \in E, \phi_p(x) = \langle p, f(x) \rangle$ , is continuous. If  $E$  is an infinite dimensional space,  $f$  may exhibit some discontinuities, as a mapping from  $E$  to  $E$ . Yet, it is half-continuous (this is a consequence of Proposition 2.4.)

**Example 2.8.** Let  $E$  be a Banach space,  $C$  a convex and compact subset of  $E$  and  $F$  a multivalued mapping from  $C$  to  $C$  with a closed graph and nonempty convex values. Suppose that for every  $x \in C$  one has  $x \notin F(x)$ , i.e.  $F$  has no fixed point. Then any selection  $f$  of  $F$  is half-continuous (Recall that  $f$  is said to be a selection of  $F$  if for every  $x \in C$ ,  $f(x) \in F(x)$ ).

To prove this result, let  $\bar{x} \in C$ . Since  $F(\bar{x})$  is convex and since, by assumption,  $\bar{x} \notin F(\bar{x})$ , Hahn-Banach theorem implies that there exists  $p \in E^*$  such that

$$(2.4) \quad \forall y \in F(\bar{x}), \langle p, y - \bar{x} \rangle > 0.$$

and in particular, since  $f$  is a selection of  $F$ ,

$$(2.5) \quad \langle p, f(\bar{x}) - \bar{x} \rangle > 0.$$

Now, if  $f$  is not half-continuous at  $\bar{x}$ , then from Equation 2.5, there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  of  $C$  converging to  $\bar{x}$ , and such that

$$(2.6) \quad \forall n \in \mathbb{N}, \langle p, f(x_n) - x_n \rangle \leq 0.$$

Let us define, for every integer  $n \in \mathbb{N}$ ,  $y_n = f(x_n)$ . Since  $f$  is a selection of  $F$ , one has  $y_n \in F(x_n)$  for every integer  $n$ . From the compactness of  $C$ , and since  $F$  has a closed graph, the sequence  $(y_n)_{n \in \mathbb{N}}$  converges (up to an extraction) to  $\bar{y} \in F(\bar{x})$ .

Passing to the limit in Equation 2.6, one obtains

$$(2.7) \quad \langle p, \bar{y} - \bar{x} \rangle \leq 0$$

which contradicts Equation 2.4, because  $\bar{y} \in F(\bar{x})$ .

**Example 2.9.** Let  $E$  be a Banach space and  $C$  be a subset of  $E$ . Consider a mapping  $f : C \rightarrow E$ , and define the two following assumptions:

(A1) For every  $x \in C$  with  $x \neq f(x)$ , there exists  $\epsilon > 0$  such that

$$(2.8) \quad \forall x' \in B(x, \epsilon) \cap C, \|f(x) - x\| > \|f(x') - f(x)\| + \|x' - x\|$$

(A2) For every  $x \in C$  with  $x \neq f(x)$ , there exists  $\epsilon > 0$  such that

$$(2.9) \quad \forall x' \in B(x, \epsilon) \cap C, \|f(x) - x\| > \|f(x') - f(x) - x' + x\|$$

Clearly, Assumption (A1) implies Assumption (A2), but the converse is not true. We now prove that each of these two Assumptions above imply half-continuity. We only have to prove that Assumption (A2) implies half-continuity. Let  $x \in C$  with  $x \neq f(x)$ . From Hahn-Banach theorem, there exists  $p \in E^*$  such that

$$(2.10) \quad \langle p, f(x) - x \rangle = \|f(x) - x\|^2$$

and

$$(2.11) \quad \|p\| = \|f(x) - x\|.$$

For every  $x' \in C$ , one has, from Equation 2.10 and from the bilinearity of scalar product

$$\langle p, f(x') - x' \rangle = \|f(x) - x\|^2 - \langle p, f(x) - f(x') - (x - x') \rangle$$

so that for every  $x' \in C \cap B(x, \epsilon)$

$$\langle p, f(x') - x' \rangle \geq \|f(x) - x\|(\|f(x) - x\| - \|f(x') - f(x) - x' + x\|) > 0$$

from Cauchy-schwarz inequality, Equation 2.11 and Equation 2.9.

**Example 2.10.** Let  $C$  be a bounded subset of  $H$ , a Hilbert space. Let  $g : C \rightarrow H$  be a continuous mapping, let  $\epsilon > 0$  and define  $f : C \rightarrow C$  be any mapping such that

$$\forall x \in C, x \neq f(x) \Rightarrow \langle f(x) - x, g(x) \rangle > \epsilon.$$

Then  $f$  is half-continuous. Indeed, let  $\bar{x} \in C$ . Since  $C$  is bounded, there exists  $n \in \mathbb{N}^*$  such that

$$(2.12) \quad \forall x \in C, \|x\| \leq n$$

From the continuity of  $g$ , there exists  $\epsilon > 0$  such that

$$(2.13) \quad \forall x \in B(\bar{x}, \epsilon) \cap C, \|g(x) - g(\bar{x})\| < \frac{\epsilon}{2n}.$$

Let define  $p = g(\bar{x})$ . Then for every  $x \in B(\bar{x}, \epsilon) \cap C$  with  $f(x) \neq x$ ,

$$(2.14) \quad \langle p, f(x) - x \rangle = \langle g(x), f(x) - x \rangle + \langle p - g(x), f(x) - x \rangle > \epsilon + \langle p - g(x), f(x) - x \rangle$$

which is strictly positive from Equation 2.13, Equation 2.12 and Cauchy-Schwarz inequality.

**Example 2.11.** Let  $E$  be a Banach space,  $C$  be subset of  $E$ , and let  $g : C \rightarrow E$  be a half-continuous mapping without fixed point such that for every  $x \in C$ , the set

$$]x, g(x)] = \{tx + (1 - t)g(x), 0 \leq t < 1\}$$

contains at least one point of  $C$ , denoted  $f(x)$ . Then the mapping  $f$  is half-continuous.

Indeed, let  $\bar{x} \in C$ . For every  $x \in C$ , let  $t(x) \in [0, 1[$  such that

$$(2.15) \quad f(x) = t(x)x + (1 - t(x))g(x).$$

Since  $g$  is half-continuous, and since  $g(\bar{x}) \neq \bar{x}$ , there exists  $\epsilon > 0$  and  $p \in E^*$  such that for every  $x \in B(\bar{x}, \epsilon) \cap C$ , one has

$$(2.16) \quad \langle p, g(x) - x \rangle > 0.$$

Then for every  $x \in B(\bar{x}, \epsilon) \cap C$ , one has

$$\langle p, f(x) - x \rangle = \langle p, t(x)x + (1 - t(x))g(x) - x \rangle = (1 - t(x))\langle p, g(x) - x \rangle > 0$$

from Equation 2.16 and 2.15, which proves that  $f$  is half-continuous.

### 3. THE MAIN FIXED POINT THEOREM

For stating our main result, recall the following definitions: for every  $C$ , a closed convex subset of a Banach space  $E$ , and for every  $x \in C$ , the Clarke tangent cone  $T_C(x)$  (see [4] p. 86) can be defined by  $T_C(x) := cl\{\lambda(y - x), \lambda \geq 0, y \in C\}$  where  $cl(\cdot)$  denotes the closure of a set in  $E$  (for the strong topology).

Besides, the Clarke normal cone  $N_C(x)$  (see [4] p. 86) is the negative polar cone of  $T_C(x)$ , but in the convex case it can be written more simply  $N_C(x) := \{p \in E^*, \forall y \in C, \langle p, y - x \rangle \leq 0\}$ .

#### 3.1. Statement of the result.

**Theorem 3.1.** *Let  $C$  be a compact and convex subset of  $E$ , a Banach space, and let  $f : C \rightarrow C$  be a half-continuous mapping. Then  $Fix(f) \neq \emptyset$ .*

*Proof.* Suppose, on the contrary, that  $Fix(f) = \emptyset$ . For every  $x \in C$ , let us define a multivalued mapping  $\Phi$  from  $C$  to  $E^*$  by

$$\Phi(x) = \{p \in E^*, \exists \epsilon > 0, \forall x' \in B(x, \epsilon) \cap C, f(x') \neq x' \Rightarrow \langle p, f(x') - x' \rangle > 0\}$$

for every  $x \in C$ . First of all, since  $f$  is a half-continuous mapping,  $\Phi$  has nonempty values. Secondly,  $\Phi$  has convex values: indeed, if  $p \in \Phi(x)$ ,  $q \in \Phi(x)$  and  $\lambda \in [0, 1]$ , then there exists  $\epsilon > 0$  and  $\epsilon' > 0$  such that

$$\forall x' \in B(x, \epsilon) \cap C, f(x') \neq x' \Rightarrow \langle p, f(x') - x' \rangle > 0$$

and

$$\forall x' \in B(x, \epsilon') \cap C, f(x') \neq x' \Rightarrow \langle q, f(x') - x' \rangle > 0$$

so that one has

$$\forall x' \in B(x, \min\{\epsilon, \epsilon'\}) \cap C, f(x') \neq x' \Rightarrow \langle \lambda p + (1 - \lambda)q, f(x') - x' \rangle > 0,$$

which proves  $\lambda p + (1 - \lambda)q \in \Phi(x)$ . Thirdly,  $\Phi$  has open preimages: indeed, let  $\bar{x} \in C$  and  $p \in \Phi(\bar{x})$ . We want to prove that the set  $\mathcal{O} = \{x \in C, p \in \Phi(x)\}$  is an open neighborhood of  $\bar{x}$ , where  $\mathcal{O}$  can also be written

$$\mathcal{O} = \{x \in C, \exists \epsilon > 0, \forall x' \in B(x, \epsilon) \cap C, f(x') \neq x' \Rightarrow \langle p, f(x') - x' \rangle > 0\}.$$

But this is clear, since one has  $B(\bar{x}, \frac{\epsilon}{2}) \subset \mathcal{O}$ .

Now, recall that from Browder selection theorem (see [2]), if  $X$  is a paracompact Hausdorff topological space,  $Z$  a topological vector space,  $T : X \rightarrow Z$  a multivalued mapping having nonempty convex values and open preimages, then  $T$  has a continuous selection. Here,  $\Phi$  satisfies the assumptions of Browder selection theorem. So, there exists a continuous selection  $\phi : C \rightarrow E^*$  of  $\Phi$ .

Besides, from Cwiszewski and Kryszewski (see [9], Theorem 2.1, p.713), if  $C$  is a compact and convex subset of  $E$  and  $\phi : C \rightarrow E^*$  is a continuous mapping, then there exists  $x \in C$  such that  $\phi(x) \in N_C(x)$ .

Thus, this theorem applied to  $\phi$  implies that there exists a solution  $\bar{x} \in C$  of the following variational inequality problem

$$(3.1) \quad \phi(\bar{x}) \in N_C(\bar{x}).$$

Now, from Equation 3.1, from  $f(\bar{x}) \in C$  and from the definition of the normal cone in the convex case, one has

$$(3.2) \quad \langle \phi(\bar{x}), f(\bar{x}) - \bar{x} \rangle \leq 0.$$



But from the definition of  $\Phi(\bar{x})$  and from  $\phi(\bar{x}) \in \Phi(\bar{x})$ , one has  $\langle \phi(\bar{x}), f(\bar{x}) - \bar{x} \rangle > 0$ , a contradiction with Equation 3.2. This ends the proof of Theorem 3.1.  $\square$

**3.2. A multivalued version of Theorem 3.1.** In many applications, one needs fixed point theorems for multivalued mappings. In fact, Theorem 3.1 admits an immediate multivalued generalization.

**Theorem 3.2.** *Let  $C$  be a convex and compact subset of a Banach space  $E$ , and let  $F$  be a multivalued mapping from  $C$  to  $C$ , half-continuous in the following sense: for every  $x \in C$  such that  $x \notin F(x)$ , there exists  $p \in E^*$  and  $\epsilon > 0$  with*

$$\forall x' \in B(x, \epsilon) \cap C, x' \notin F(x') \Rightarrow \forall y' \in F(x'), \langle p, y' - x' \rangle > 0.$$

*Then  $\text{Fix}(F) \neq \emptyset$ .*

*Proof.* If  $F$  has no fixed point, then consider any selection  $f$  of  $F$ : it is clearly half-continuous, and so, from Theorem 3.1, it admits a fixed point, a contradiction.  $\square$

*Remark 3.3.* It is easy to prove that a multivalued mapping with a closed graph and nonempty convex values satisfies the assumption of Theorem 3.2, so that Kakutani theorem is a consequence of this theorem. See the next subsection for another explicit proof of Kakutani fixed point theorem from Theorem 3.1.

**3.3. A discontinuous version of a theorem of Fan.** We now provide the following generalization of a theorem due to Fan (see, for example, [7] p. 146), which has treated the continuous case:

**Theorem 3.4.** *Let  $E$  be a Banach space,  $C$  be subset of  $E$ , and let  $g : C \rightarrow E$  be a half-continuous mapping such that for every  $x \in C$  with  $x \neq g(x)$ , the set*

$$[x, g(x)] = \{tx + (1-t)g(x), 0 \leq t < 1\}$$

*contains at least one point of  $C$ . Then  $g$  admits a fixed-point.*

*Proof.* Suppose that  $g$  does not admit any fixed point. For every  $x \in C$ , let  $f(x) \in [x, g(x)] \cap C$ . From example 2.11,  $f$  is half-continuous from  $C$  to  $C$ , and so admits a fixed point from Theorem 3.1. This is a contradiction.  $\square$

**3.4. Some byproducts: Schauder, Brouwer, Kakutani fixed point theorems.** A straightforward consequence of our main result is the following version of Schauder fixed point theorem, which also implies Brouwer fixed point theorem:

**Theorem 3.5.** *Let  $C$  be a compact and convex subset of  $E$ , a Banach space, and let  $f : C \rightarrow C$  be a continuous mapping. Then  $\text{Fix}(f) \neq \emptyset$ .*

Similarly, Theorem 3.1 implies Kakutani's theorem (which can also be derived from Theorem 3.2):

**Theorem 3.6.** *Let  $C$  be a compact and convex subset of  $E$ , a Banach space, and let  $F : C \rightarrow 2^C$  be a multivalued mapping with a closed graph and with nonempty convex values. Then  $\text{Fix}(F) \neq \emptyset$ .*

*Proof.* If  $\text{Fix}(F) = \emptyset$  then from example 2.8, any selection  $f$  of  $F$  is half-continuous, and from Theorem 3.1,  $f$  admits a fixed point, a contradiction. Thus  $\text{Fix}(F) \neq \emptyset$ .  $\square$

**3.5. An answer to an open question.** A corollary of Theorem 3.1 is the following theorem:

**Theorem 3.7.** *Let  $C$  be a compact and convex subset of  $\mathbb{R}^n$ , and let  $f : C \rightarrow C$ . Assume the following property is true:*

*(P) for every  $x \in C$  such that  $f(x) \neq x$ , there exists  $V_x$ , an open neighborhood of  $x$  in  $C$  such that for every  $u$  and  $v$  in  $V_x$ ,  $\langle f(u) - u, f(v) - v \rangle \geq 0$ .*

*Then, there exists  $x \in C$  such that  $f(x) = x$ .*

*Remark 3.8.* Herings et al. ([10]) proved a theorem similar to Theorem 3.7, but in the restrictive case where  $C$  is a polyhedron of  $\mathbb{R}^n$ . They introduced property (P) and called it "locally gross direction preserving". In [10], one can read: "whether locally gross direction preserving is sufficient to guarantee the existence of a fixed point on an arbitrary convex and compact set (of  $\mathbb{R}^n$ ) is still an open question." Thus, the theorem above clearly answers to this question.

*Proof.* To prove the corollary, one only has to prove that Property (P) implies the half-continuity of  $f$ , and to apply Theorem 3.1. Let suppose that Property (P) is true. Let  $x \in C$  such that  $f(x) \neq x$ , and let  $V_x$  be an open neighborhood of  $x$  as defined by Property P. Let  $\{f(x_1) - x_1, \dots, f(x_k) - x_k\}$  be a basis of the vector space  $F := \text{span}\{f(y) - y, y \in V_x\}$ , where  $x_1, \dots, x_k$  are in  $V_x$ . Then define

$$p = \sum_{i=1}^k (f(x_i) - x_i).$$

Let  $x' \in V_x$  such that  $f(x') \neq x'$ . One clearly have

$$(3.3) \quad \langle p, f(x') - x' \rangle \geq 0$$

from property (P) and from the definition of  $p$ .

Besides, Inequation 3.3 is an equality if and only if one has

$$\forall i = 1, \dots, k, \langle f(x') - x', f(x_i) - x_i \rangle = 0.$$

It implies  $f(x') - x' \in F^\perp \cap F = \{0\}$ , a contradiction with the assumption that  $f(x') \neq x'$ . Thus, Inequation 3.3 is strict, which proves that  $f$  is half-continuous.  $\square$

#### 4. APPROXIMATE FIXED POINT THEOREMS

Several papers have been extended Brouwer fixed point theorem to noncontinuous mappings. A classical approach has been to prove the existence of an approximate fixed point for some classes of almost continuous mappings (see, for example, [8], [5], [6] and [3]). The most common notion of approximate fixed point is the following:

**Definition 4.1.** Let  $f : C \rightarrow E$ , where  $C$  is a subset of a normed linear space  $E$ . Let  $r > 0$ . One says that  $f$  admits a  $r$ -fixed point if:

$$\forall \epsilon > 0, \exists x_\epsilon \in C, \|f(x_\epsilon) - x_\epsilon\| \leq r + \epsilon.$$

Let us then introduce the two following notions of almost continuity (these notions can be found, for example, in [8] for the first notion and [3] for the second notion):

**Definition 4.2.** Let  $f : C \rightarrow E$ , where  $C$  is a subset of a normed linear space  $E$ . Let  $r > 0$ . One says that  $f$  is strongly<sup>3</sup>  $r$ -continuous if:

$$\forall x \in C, \exists \delta > 0, \forall (y, z) \in C \cap B(x, \delta) \times C \cap B(x, \delta), \|f(y) - f(z)\| \leq r.$$

One says that  $f$  is  $r$ -continuous if:

$$\forall x \in C, \forall \epsilon > 0, \exists \delta > 0, \forall y \in C \cap B(x, \delta), \|f(x) - f(y)\| < r + \epsilon.$$

*Remark 4.3.* Clearly, if  $f$  is strongly  $r$ -continuous, then it is  $r$ -continuous (take  $z = x$  in the first definition).

It is a well known result that every mapping  $f : C \rightarrow C$ , where  $C$  is a convex and compact subset of a normed linear space, admits a  $r$ -fixed point if  $f$  is strongly  $r$ -continuous (see [8]) and admits a  $2r$ -fixed point if  $f$  is  $r$ -continuous (see [3]). These two results are based on an approximation of  $f$  by a continuous mapping.

An easy consequence of Theorem 3.1 is the following theorem, which improves the result of [3], and unifies the two results in [3] and [8]:

**Theorem 4.4.** *Every  $r$ -continuous mapping  $f : C \rightarrow C$ , where  $C$  is a convex and compact subset of a Banach space, admits a  $r$ -fixed point.*

*Proof.* Suppose that  $f$  does not admit a  $r$ -fixed point, which can be written:

$$(4.1) \quad \exists \epsilon > 0, \forall x \in C, \|f(x) - x\| > r + \epsilon.$$

Let  $x \in C$  and let  $\epsilon > 0$  as above. Writing the  $r$ -continuity of  $f$  using  $\epsilon' = \frac{\epsilon}{3}$ , we obtains that there exists  $\delta > 0$  (now defined) such that:

$$(4.2) \quad \forall y \in C, \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < r + \frac{\epsilon}{3}.$$

Let then defined  $\delta' = \min\{\delta, \frac{\epsilon}{3}\} > 0$ . One has for every  $y \in B(x, \delta') \cap C$

$$(4.3) \quad \|x - y\| + \|f(x) - f(y)\| \leq \|x - f(x)\|,$$

from Inequalities 4.1 and 4.2. Thus, from Example 2.9, the mapping  $f$  is half-continuous (since it satisfies property (A1)). Thus, From Theorem 3.1,  $f$  admits a fixed point, a contradiction with the first assumption. Thus,  $f$  admits a  $r$ -fixed point.  $\square$

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<sup>3</sup>In [8], such a mapping  $f$  is simply called " $r$ -continuous". We add "strongly" to distinguish this notion from the second here after, which is a weaker notion.

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